

On tree amplitudes in gauge theory and gravity

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ABSTRACT: The BCFW recursion relations provide a powerful way to compute tree amplitudes in gauge theories and gravity, but only hold if some amplitudes vanish when two of the momenta are taken to infinity in a particular complex direction. This is a very surprising property, since individual Feynman diagrams all diverge at infinite momentum. In this paper we give a simple physical understanding of amplitudes in this limit, which corresponds to a hard particle with (complex) light-like momentum moving in a soft background, and can be conveniently studied using the background field method exploiting background light-cone gauge. An important role is played by enhanced spin symmetries at infinite momentum—a single copy of a “Lorentz” group for gauge theory and two copies for gravity—which together with Ward identities give a systematic expansion for amplitudes at large momentum. We use this to study tree amplitudes in a wide variety of theories, and in particular demonstrate that certain pure gauge and gravity amplitudes do vanish at infinity. Thus the BCFW recursion relations can be used to compute completely general gluon and graviton tree amplitudes in any number of dimensions. We briefly comment on the implications of these results for computing massive 4D amplitudes by KK reduction, as well understanding the unexpected cancelations that have recently been found in loop-level gravity amplitudes.

KEYWORDS: Field Theories in Higher Dimensions, Gauge Symmetry, Classical Theories of Gravity, Gauge-gravity correspondence.

Contents

1. Introduction	1
1.1 BCFW Redux	2
1.2 Surprising behavior of $M(z)$ as $z \rightarrow \infty$	4
2. Understanding $M(z \rightarrow \infty)$ in YM and GR	5
2.1 Scalar QED amplitudes	6
2.2 Dominant large z behavior	8
2.3 Scalar-Yang-Mills amplitudes	9
2.4 Gluon amplitudes and enhanced spin symmetry as $z \rightarrow \infty$	10
2.5 Scalar-graviton amplitudes	12
2.6 Photon-graviton amplitudes	13
2.7 Amplitudes in general relativity	14
3. Discussion	17

1. Introduction

The textbook formulation of perturbative QFT as an expansion in Feynman diagrams includes an enormous amount of unphysical off-shell structure. This is particularly true of Yang-Mills theories and General Relativity, where the gauge and diffeomorphism redundancies are introduced to make Lorentz invariance and locality manifest. While Lorentz invariance is very likely an exact property of Nature, non-perturbative gravity makes it impossible to define off-shell local observables, and therefore locality is a more suspect notion at a fundamental level. It would therefore be interesting to find a different formulation of QFT not relying so heavily on manifest locality. Such a formulation might allow for a more natural inclusion of gravity, much like the non-manifestly deterministic least action formulation of classical mechanics generalizes more naturally to quantum mechanics than Newton's laws. There is a more down-to-earth reason for suspecting that another formulation of QFT exists: on-shell gauge and gravity amplitudes, particularly for many external legs, receive contributions from a huge number of Feynman diagrams, but extensive cancellations take place and the final results are strikingly simple, exhibiting regularities that are invisible in the diagrammatic expansion [1–3]. The simplicity of the final answer suggests that there should be another way of computing the amplitudes more directly.

1.1 BCFW Redux

For tree amplitudes, a huge step in this direction was taken by Britto, Cachazo, Feng [4] and further clarified with Witten [5]. Their work was an outgrowth of Witten’s twistor formulation of four-dimensional Yang-Mills [6, 7], which is crucially tied to 4D physics. Indeed, the great simplicity of maximal helicity violating amplitudes is due to their close connection to self-dual solutions of the Yang-Mills equations of motion [8, 9], which is very special to four dimensions. But the BCFW ideas do not rely on twistors or the spinor-helicity formalism, and are instead a general property of QFT in any number of dimensions, as we now review.

Consider the n -point amplitude $M(p_i, h_i)$ for massless particles with h_i “helicities” in a general number D of spacetime dimensions. When we consider gauge theory, we will define $M(p_i, h_i)$ such that the color factors are already stripped away. We will also suppress the trivial overall multiplicative coupling constant dependence. The key idea is to pick two external momenta p_j, p_k , and to analytically continue these momenta keeping them on-shell and maintaining momentum conservation. Specifically, BCFW take

$$p_j \rightarrow p_j(z) = p_j + qz \quad \text{and} \quad p_k \rightarrow p_k(z) = p_k - qz \tag{1.1}$$

where we must have $q \cdot p_{j,k} = 0$, $q^2 = 0$. This is impossible for real q , but possible for complex q . To be explicit, choose a Lorentz frame where p_j, p_k are back to back with equal energy and use units where that energy is 1. Then, we can choose

$$p_j = (1, 1, 0, 0; 0, \dots, 0), \quad p_k = (1, -1, 0, 0; 0, \dots, 0), \quad q = (0, 0, 1, i; 0, \dots, 0) \tag{1.2}$$

Note that this deformation only makes sense for $D \geq 4$.

What about the polarization tensors? Note that for gauge theory in a covariant gauge, $q = \epsilon_1^- = \epsilon_2^+$. This makes it natural to use a $+, -, T$ basis for spin 1 polarization vectors where

$$\epsilon_j^- = \epsilon_k^+ = q, \quad \epsilon_j^+ = \epsilon_k^- = q^*, \quad \epsilon_T = (0, 0, 0, 0, \dots, 1, \dots, 0) \tag{1.3}$$

with $D-4$ different ϵ_T forming a basis in the transverse directions. When the momenta are deformed, the polarization vectors must also change to stay orthogonal to their associated momenta and maintain their inner products. This requires

$$\epsilon_j^-(z) = \epsilon_k^+(z) = q, \quad \epsilon_j^+(z) = q^* - zp_k, \quad \epsilon_k^-(z) = q^* + zp_j, \quad \epsilon_T(z) = (0, 0, 0, 0, \dots, 1, \dots, 0) \tag{1.4}$$

Alternatively, we can keep all the momenta and polarization vectors real but imagine that we are working in $SO(D-2, 2)$ signature; this point of view will allow us to avoid subtleties when we take the complex conjugates of field derivatives. Graviton polarization tensors are simply symmetric, traceless products of these gauge polarization vectors. A general product of polarization tensors including the antisymmetric and trace parts gives amplitudes including a dilaton and antisymmetric two index tensor field.

With this deformation, $M(p_i, h_i) \rightarrow M(z)$ becomes a function of z . At tree level, $M(z)$ has an extremely simple analytic structure — it only has simple poles. This follows

from a straightforward consideration of Feynman diagrams, as all singularities come from propagators, which are simply

$$\frac{1}{P_J(z)^2} = \frac{1}{(\sum_{i \in J} p_i)^2} \tag{1.5}$$

where J is some subset of the n momenta. Since $p_j(z) + p_k(z)$ is independent of z , this only has non-trivial z dependence when only one of $p_j(z)$ or $p_k(z)$ are included in J . Without loss of generality we take $j \in J$, in which case we have

$$\frac{1}{P_J(0)^2 - 2zq \cdot P_J}. \tag{1.6}$$

This shows that all singularities are simple poles located at $z_J = P_J(0)^2 / (2q \cdot P_J)$. Furthermore, the residue at these poles has a very simple interpretation as a product of lower amplitudes:

$$\text{res}M(z \rightarrow z_J) = \sum_h M(i \in J, p_i(z_J), h_i; -P_J(z_J), h) \times M(i \notin J, p_i(z_J), h_i; P_J(z_J), -h). \tag{1.7}$$

where we have a sum over helicities for the usual reason, guaranteed by unitarity, that the numerator of the propagator can be replaced by the polarization sum on shell.

So far everything has been kinematical and true for an arbitrary theory. What is remarkable is that for certain amplitudes in some theories, $M(z \rightarrow \infty)$ vanishes. Now, meromorphic functions that vanish at infinity are completely characterized by their poles; if $M(z \rightarrow \infty) = 0$, we have $0 = \int dz/z M(z) = M(0) + \text{residues}$, and this gives us the BCFW recursion relation

$$M(0) = \sum_{J,h} M(i \in J, p_i(z_J); h_i, -P_J(z_J), h) \frac{1}{P_J^2} M(i \notin J, p_i(z_J), h_i; P_J(z_J), -h) \tag{1.8}$$

where h indicates a possible internal helicity. The lower amplitudes are on-shell (in complexified momentum space), because all the momenta are on shell though evaluated at a complex $z = z_J$. These recursion relations produce a higher-point amplitude by sewing together lower-point on shell amplitudes.

Of course the strategy of determining amplitudes directly from their singularities is a familiar and central theme of the S-matrix program. However, the old ideas were largely restricted to $2 \rightarrow 2$ scattering and the complexification of the Mandelstam s, t, u variables, and the generalization to higher-point amplitudes was not clear. Over the past twenty years, S-matrix ideas have had a resurgence, as it has become increasingly clear that they provide powerful methods for computing field theory amplitudes, for instance as in the unitarity methods of Bern, Dixon, Dunbar and Kosower [10]. The BCFW recursion relations are another step in this direction. Indeed, the BCFW deformation of momenta can be viewed as a correct general procedure for complexifying on-shell momenta and, at least at tree level, the recursion relations beautifully fulfil the S-matrix dream of dealing directly with on-shell amplitudes without reference to an off-shell Lagrangian.

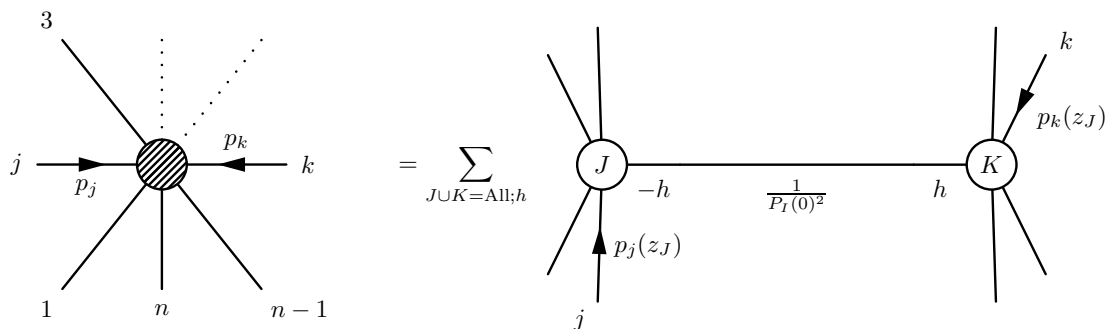


Figure 1: The BCFW recursion relation computes an n -point amplitude by sewing together lower-point amplitudes with (complex) on-shell momenta.

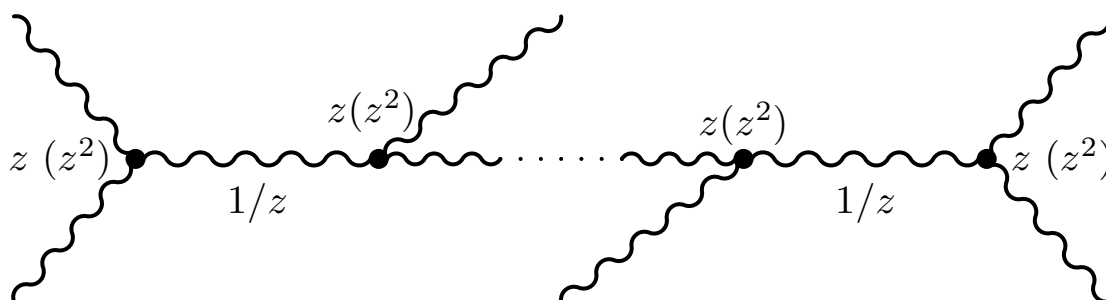


Figure 2: Contributions to the analytically continued amplitudes $M(z)$ from individual Feynman diagrams diverge as $z \rightarrow \infty$ for gauge theories and gravity. This is due to the vertices that grow as z, z^2 for gauge theory and gravity, which overcompensate for the $1/z$ scaling of propagators.

1.2 Surprising behavior of $M(z)$ as $z \rightarrow \infty$

In order to derive these recursion relations, it was necessary to assume $M(z \rightarrow \infty)$ vanishes. But this is far from obvious, and is not even true for every amplitude in a general theory. Indeed, naively it is never true! Consider for instance ϕ^4 theory, here the $2 \rightarrow 2$ amplitude is momentum independent and hence z independent. With more external lines there are propagators that as we have seen above fall as $1/z$, however, there is always a diagram with the ϕ^4 interaction involving j, k and two other lines, with each of the lines separately attaching to two separate sets of external states. The large momentum does not flow through any of the propagators, so general amplitudes go to a constant as $z \rightarrow \infty$

$$M^{\phi^4}(z) \rightarrow z^0 \tag{1.9}$$

The situation seems even worse with gauge theory and gravity, where there are momentum dependent vertices that scale as z in gauge theory and z^2 in gravity, and so we might expect that $M(z)$ diverges as $z \rightarrow \infty$. Indeed, naively in gauge theory

$$M_{\text{naive}}^{-+}(z) \rightarrow z, \quad M_{\text{naive}}^{--/++}(z) \rightarrow z^2, \quad M_{\text{naive}}^{+-}(z) \rightarrow z^3, \tag{1.10}$$

where the + and - signs represent the different gauge boson helicity states, and different powers of z result from the growth of some of these polarization vectors as $z \rightarrow \infty$. In gravity the naive divergence grows as power of the number of external legs, for example

$$M_{\text{naive}}^{-,++}(z) \rightarrow z^{n-1} \quad \dots, \quad M_{\text{naive}}^{+,--}(z) \rightarrow z^{n+3}. \quad (1.11)$$

Nonetheless, at least for some h_j, h_k , the amplitudes *do* vanish as $z \rightarrow \infty$! Even when they do not, they often diverge more mildly than the naive expectation; this neatly encapsulates the heavy cancellations that take place in the explicit evaluation of Feynman digrams. For instance for 4D gauge theories, BCFW showed that

$$M^{-+}(z), M^{--}(z), M^{++}(z) \rightarrow \frac{1}{z} \quad \text{and} \quad M^{+-} \rightarrow z^3. \quad (1.12)$$

BCFW gave a simple diagrammatic proof for $(-+)$, whereas the $(--)$ and $(++)$ cases needed a different argument based on MHV diagrams and the CSW recursion relations, which are very special to $D = 4$. Note that for the recursion relations to hold, it is not necessary for $M(z)$ to vanish for all h_{jk} helicities. For gauge theory it suffices to have e.g. $M_{-h}(z \rightarrow \infty)$ all vanish, since for any amplitude, we can always choose q to co-incide with ϵ_1^- ; for gravity it suffices for $M_{-h}(z \rightarrow \infty)$ to vanish for the same reason.

In 4D gravity, surprisingly good behavior for amplitudes was first observed by [11, 12]. Cachazo et. al. then showed in beautiful papers [13, 14] that

$$M^{--,++} \rightarrow 1/z^2 \quad \text{and} \quad M^{+,--} \rightarrow z^6 \quad (1.13)$$

Their analysis for the $(--,++)$ case involved intricate diagrammatic and combinatorial recursion arguments, and the $(--,--)$, $(++,++)$ and $(++,--)$ cases were only controlled for MHV amplitudes. Subsequently, Bern et. al. showed that the $(++,--)$ scaling holds in general, and found the general scaling for all helicity combinations up to 10 external legs [15, 16]. Note that the z scaling conforms to the famous KLT pattern $M_{\text{grav}} \sim M_{\text{gauge}} \times M_{\text{gauge}}$ [17], (though KLT only controls these amplitudes for $2 \rightarrow 2$ scattering while for more legs, term by term the amplitudes are nearly as divergent as with standard Feynman diagrams). It is remarkable that, far from being uncontrollably divergent, certain gravity amplitudes are even better behaved at infinity than their gauge counterparts, which are in turn better behaved than the simplest scalar field theories!

There is clearly simplicity and a pattern to $M(z \rightarrow \infty)$ in gauge theory and gravity. What is known so far in generality is restricted to four dimensions, and the techniques used to understand the large z behavior differ from case to case and do not illuminate this pattern, or tell us what to expect for general theories in any number of dimensions. In this paper, we will develop a more transparent, physical understanding of the large z behavior that is valid in any number of dimensions; apart from a clearer understanding of the physics this also immediately generalizes the BCFW recursion relations to gauge and gravity amplitudes in any number of dimensions.

2. Understanding $M(z \rightarrow \infty)$ in YM and GR

We begin by observing that as $z \rightarrow \infty$, the momenta $p_{j,k}$ tend to $\pm zq$, and if we think of one as ingoing and the other as outgoing, this is simply a limit where a hard light-like

particle is shooting through a soft background. For *real* momenta, this is the very familiar eikonal limit — indeed the soft collinear effective theory [18] provides a natural formalism for studying physics in this regime, though we won't make any use of the machinery of this subject in our analysis. Intuitively, a highly boosted particle will not be “much” scattered by the background, and its helicity should be conserved. This is not precisely our situation because our hard light-like momentum is complex (or equivalently real but in $(D - 2, 2)$ signature). We will see that we can understand the behavior of the amplitudes at large z as an expansion in $1/z$, that both quantifies the intuition for real momenta and can be used to understand the scalings for the complex momenta of interest. Since we only care about the z dependence of the amplitudes, all the soft physics can be absorbed into determining some classical background, and the single hard line can be studied by looking at quadratic fluctuations about this background. Another natural approach would be to use the worldline formalism for a particle propagating in a background field [20], but we will see that the standard field theory techniques are already very simple.

We proceed to study the large z behavior of amplitudes in theories of increasing complexity; scalar QED, scalar Yang-Mills, Yang-Mills, gravity coupled to a scalar, to a photon, and finally gravity itself.

2.1 Scalar QED amplitudes

We start with scalar QED as a simple warm-up. We will dwell on a number of issues at greater length in this subsection where they can be explored in the simplest setting, and abbreviate the analogous discussion in the subsequent subsections.

Consider amplitudes M_n with exactly two external scalar lines and n external photons. We choose to analytically continue the momenta corresponding to the scalar lines. The scalar Lagrangian is

$$L = D_\mu \phi^* D^\mu \phi. \tag{2.1}$$

where we view A_μ as a background field in which the highly boosted scalar particle propagates. Note that naively $M(z) \rightarrow z$ for large z , since the scalar-scalar-photon vertex has a momentum that scales as z . We will see that however that $M_n(z)$ is in fact much better behaved.

In considering the amplitude for large momentum, we immediately run into the problem that ‘large’ momenta for ϕ is not a gauge-invariant. The natural way to deal with this issue is to perform a field re-definition (as in SCET [19]), stripping off a Wilson line from the field

$$\phi(x) = W_n(x) \tilde{\phi}(x), \quad \text{where} \quad W_n(x) = \exp \left(i \int_{-\infty}^0 d\lambda n_\mu A^\mu(x + \lambda n) \right) \tag{2.2}$$

and n^μ determines the direction of the Wilson line stretching from the point $-\infty$ to x . Since $\tilde{\phi}(x)$ is gauge invariant, its ordinary derivative gives a gauge invariant definition of momentum. The Lagrangian becomes

$$L = (W_n \partial_\mu \tilde{\phi} + D_\mu W_n \tilde{\phi})^* (W_n \partial^\mu \tilde{\phi} + D^\mu W_n \tilde{\phi}). \tag{2.3}$$

and the only terms that grow as $z \rightarrow \infty$ are the cross terms such as

$$\partial_\mu \tilde{\phi}^* \tilde{\phi} W_n^* D^\mu W_n \rightarrow iz q_\mu \tilde{\phi}^* \tilde{\phi} W_n^* D^\mu W_n. \tag{2.4}$$

Now, $W_n^* D_\mu W_n$ is gauge invariant and a trivial computation shows

$$q_\mu W_n^* D^\mu W_n = i \int_{-\infty}^0 d\lambda q_\mu F^{\mu\nu}(x + n\lambda) n_\nu \tag{2.5}$$

Choosing $n_\mu = q_\mu$, corresponding to the Wilson line pointing in the light-like direction of the large momentum q_μ itself, this combination vanishes due to the antisymmetry of $F^{\mu\nu}$, showing in a gauge-invariant way that there are no physical $O(z)$ vertices in this theory.

Of course, there is a much simpler way of eliminating gauge redundancy and working with gauge invariant quantities—we can simply choose a gauge! The only $O(z)$ interactions from the original lagrangian involve $q \cdot A$ terms, so the natural choice is q -light cone gauge, with

$$q \cdot A = A_- = 0. \tag{2.6}$$

Indeed, the gauge invariant $-iW_q^* D_\mu W_q$ we encountered in the previous paragraph is nothing but A_μ itself in this light-cone gauge.

Although the complex q may make this gauge choice seem unusual, one can simply view it as a light cone gauge in $SO(D - 2, 2)$ signature. The utility of q -light cone gauge for gauge theory computations was recognized by Chalmers and Siegel in [21], where it was dubbed “space-cone” gauge. Vaman and Yao [22] made use of this gauge to give an understanding of the BCFW rules for gauge theory. In our discussion of Yang-Mills theory and especially gravity in the next subsections, we will simply go to light-cone gauge rather than give the analog of the explicitly gauge invariant description in terms of Wilson lines as in the above.

We have seen that there are no interaction vertices at $O(z)$. The scalar propagator is proportional to $1/z$, so all diagrams with at least one scalar propagator vanish as $z \rightarrow \infty$. Of course, due to the four-point interaction vertex, $M_2(z)$ goes to a constant at large z , however, all n -point amplitudes with $n > 2$ photons must have at least one scalar propagator, and so we conclude

$$M_2(z) \rightarrow z^0 \quad \text{and} \quad M_{n>2}(z) \rightarrow \frac{1}{z}, \tag{2.7}$$

In fact clearly, for large n there are more propagators and the amplitudes are suppressed by higher powers of z . Thus the BCFW Recursion Relations apply to scalar QED with two external scalars and at least three external photons.

It is worth noting that the better behavior of the amplitudes at large z is not merely a consequence of gauge invariance. The addition of higher dimension gauge invariant operators, for example the operator $D_\mu \phi^* D_\nu \phi F_\alpha^\mu F^{\nu\alpha}$, leads to vertices with positive powers of z . The good behavior of $M(z \rightarrow \infty)$ is thus a special feature of the two-derivative Lagrangian neglecting any higher-dimension operators.

2.2 Dominant large z behavior

Our arguments above for eliminating the $O(z)$ vertices either using Wilson lines or going to light-cone gauge were a little too quick; there is a subtlety that is irrelevant for scalar QED but will be important in the rest of the examples, and will allow us to isolate the dominant large z amplitude in all cases. Consider again our argument that $q_\mu W_n^* D^\mu W_n$ vanishes for $n_\mu = q_\mu$. It is true that the integrand $q_\mu F^{\mu\nu}(x + n\lambda)n_\nu$ vanishes as $n_\mu \rightarrow q_\mu$, but there is also a semi-infinite integral over λ , so one might have a $0 \times \infty$ ambiguity. Indeed, simply going to momentum space we can perform the λ integral and find

$$[q_\mu W_n^* D^\mu W_n](p) = \frac{q_\mu F^{\mu\nu}(p)n_\nu}{p \cdot n} = \frac{(A(p) \cdot q)(p \cdot n) - (A(p) \cdot n)(p \cdot q)}{p \cdot n} \quad (2.8)$$

As long as $p \cdot q \neq 0$, we can take $n_\mu \rightarrow q_\mu$ and this vanishes as expected. But if $p \cdot q = 0$, then we have a problem — as $n \rightarrow q$ there is a $0/0$ cancelation and we find

$$\lim_{n \rightarrow q} [q_\mu W_n^* D^\mu W_n](p) = A(p) \cdot q \quad (2.9)$$

goes to a constant.

This subtlety also shows up in going to light-cone gauge, and indeed obstructs making this gauge choice. In order to choose light cone gauge, in momentum space we need to find a $\Lambda(p)$ so that

$$q^\mu A_\mu(p) + iq^\mu p_\mu \Lambda(p) = 0, \quad (2.10)$$

but this is impossible if $p \cdot q = 0$ unless the gauge transformation becomes singular $\Lambda(p) \rightarrow \infty$.

We conclude that if there is a component of the background field carrying a momentum p such that $p \cdot q = 0$, there *is* a physical $O(z)$ vertex. In scalar QED, the photons are non-interacting so the background field is a sum of the external plane waves, and the background field momenta are just the external momenta. In non-Abelian theories there are self-interactions, and the possible components of background field momenta are simply sums of subsets of the external soft particle momenta. Thus generic momenta will indeed have $p \cdot q \neq 0$ and the subtlety above is irrelevant. But the sum over *all* the external momenta must equal $-(p_j + p_k)$, which is orthogonal to q . Therefore, there is a unique set of diagrams where this subtlety is relevant — those diagrams where the two external analytically continued lines meet in a three point vertex with a single background field, which then connects to *all* remaining external fields.

In scalar QED this never occurs, because the only such diagram would only include one photon. However, this unique class of diagrams do occur and will be of importance in non-Abelian gauge theory and even more so in gravity. The z scaling of these diagrams is the naive one corresponding to the number of momenta in the vertex — up to $O(z)$ for gauge theory and $O(z^2)$ for gravity, and therefore these unique diagrams dominate the amplitude at large z .

We can rephrase this discussion directly in the q -lightcone gauge. Here, the only singularities occur when we eliminate the “+” components of the gauge or gravity fields, for

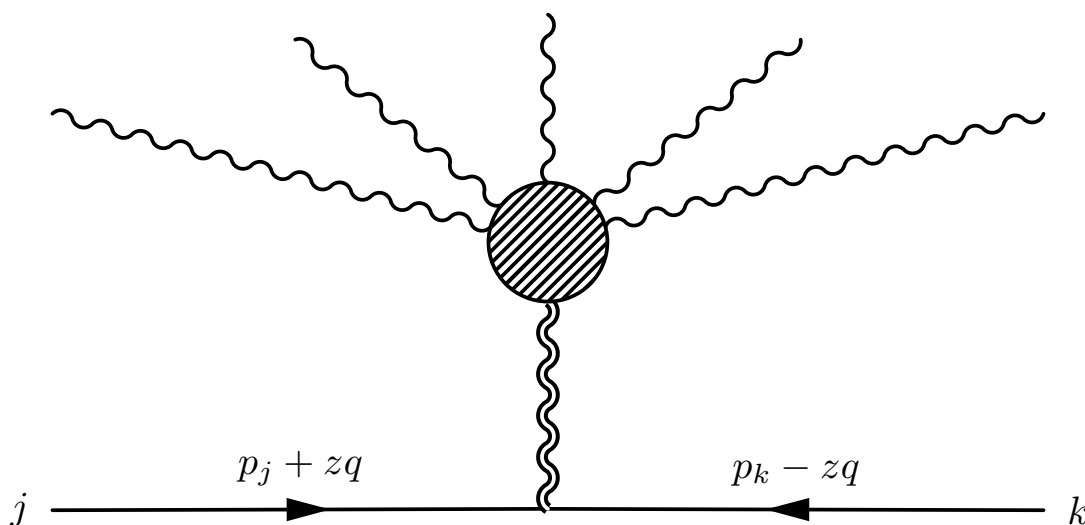


Figure 3: The unique set of diagrams, for which light-cone gauge is singular, and which dominate large z amplitudes.

instance in gauge theory we have $\partial_- A_+ - \partial_i A^i = 0$, so that $A_+(p)$ has a $1/p_-$ singularity. As we have seen, there is a unique set of diagrams where p_- vanishes for an internal background line. Since none of the other diagrams have this singularity and the full amplitude is gauge invariant, we conclude that the $1/p_-$ factors are always cancelled by p_- factors in vertices to leave a non-singular result from this unique class of diagrams alone. But, since the light-cone gauge choice is non-singular for the other diagrams where the leading z dependence is eliminated, we can also conclude that nothing can cancel the naive z scaling of our unique set of diagrams either. Thus we have identified the leading contributions to the amplitude at large z .

2.3 Scalar-Yang-Mills amplitudes

We move on to consider scalar Yang-Mills, where the above discussion can be seen in action in the simplest setting. In this case, the Lagrangian is

$$L = \text{tr} D_\mu \phi^* D^\mu \phi \tag{2.11}$$

The soft background field A_μ is the solution of the Yang-Mills equations of motion that non-linearly completes the sum over the plane waves corresponding to the soft external gluons (formally, to insure that only connected diagrams are summed, the soft polarization vectors should be thought of as being anti-commuting, see e.g. [9]). We will be assuming that ϕ is in the adjoint representation of the gauge group, since this will allow greater cohesion when we move on to pure Yang-Mills in the next section.

Now for the physics. We can again fix q -lightcone gauge to eliminate the $O(z)$ vertices as in scalar QED, but now, due to gluon self-interactions, we have the unique set of diagrams described above, with an arbitrary number of external gluons that only include a single

scalar-gluon vertex, where two scalars with momenta $p_j(z)$ and $p_k(z)$ meet an off-shell gluon field with momentum $-(p_j+p_k)$. Since the two external scalar lines in these diagrams couple directly through a 3-point vertex, the scalars must be adjacent in color at leading order in N_c (planar diagrams). As we argued in the previous subsection, these diagrams scale as $O(z)$.

For scalars that are not adjacent in color, the self-interactions of the gluons allow diagrams without scalar propagators to contribute, so the amplitude will be $O(1)$. So we have found

$$M_{jk \text{ adj. in color}}(z) \rightarrow z \quad \text{and} \quad M_{jk \text{ non-adj. in color}}(z) \rightarrow z^0 \tag{2.12}$$

This behavior can be readily verified in simple examples where explicit amplitudes are known, such as $2 \rightarrow 2$ scattering. Thus unlike scalar QED, there are no BCFW Recursion Relation in scalar Yang-Mills theory when the external scalar lines are analytically continued.

2.4 Gluon amplitudes and enhanced spin symmetry as $z \rightarrow \infty$

For gluon amplitudes in gauge theory, we will study the two particle amplitude $M^{\mu\nu}$ in a soft background field, where μ and ν are the Lorentz indices that will be contracted with the polarization vectors of the highly boosted particle (we suppress the color indices on the amplitude). A new feature will be the presence of an enhanced spin ‘‘Lorentz’’ symmetry which will largely control the large z behavior of the amplitude. Together with a simple application of the Ward identity, this will yield the desired large z scalings.

Expanding the gauge field $\mathcal{A}_\mu = A_\mu + a_\mu$ where A_μ is the background and a_μ the fluctuation, the quadratic Lagrangian for a_μ is

$$L = -\frac{1}{4} \text{tr} D_{[\mu} a_{\nu]} D^{[\mu} a^{\nu]} + \frac{i}{2} \text{tr} [a_\mu, a_\nu] F^{\mu\nu} \tag{2.13}$$

where D is the A -covariant derivative. As usual in the background field method, we have two types of gauge symmetry — gauge transformations of a and of A . We fix the a gauge freedom in the usual way by adding a gauge fixing term $(D_\mu a^\mu)^2$ to the lagrangian. The gauge fixed Lagrangian is

$$L = -\frac{1}{4} \text{tr} D_\mu a_\nu D^\mu a^\nu + \frac{i}{2} \text{tr} [a_\mu, a_\nu] F^{\mu\nu} \tag{2.14}$$

Note that the first term in this Lagrangian is the only one with the potentially $O(z)$ vertices, and hence dominates in the amplitude as $z \rightarrow \infty$. But this first term also enjoys an enhanced ‘‘spin’’ symmetry — a Lorentz transformation acting on the ν indices of a_ν alone. We’ll call this a spin ‘‘Lorentz’’ invariance, since the actual Lorentz invariance is explicitly broken by the non-vanishing background field even for the first term in the Lagrangian. To make this symmetry more explicit, we trivially re-label indices so that the Lagrangian is

$$L = -\frac{1}{4} \text{tr} \eta^{ab} D_\mu a_a D^\mu a_b + \frac{i}{2} \text{tr} [a_a, a_b] F^{ab} \tag{2.15}$$

As already noted, the first term dominates the large z amplitude but is spin “Lorentz” invariant, while the second term breaks the Lorentz symmetry as an antisymmetric tensor. This allows us to determine the form of M^{ab} . Since all the $O(z)$ vertices come from the first term, and only repeated use of these vertices can possibly give an amplitude that scales as z , the part of the amplitude that scales as z must also be proportional to η^{ab} . This reflects the intuitively familiar fact that the helicity of a highly boosted particle blasting through a soft background is conserved. The first contribution that breaks the “Lorentz” spin symmetry arises from a single insertion of vertices coming from the second term in the Lagrangian, and must be antisymmetric in (ab) , just as F^{ab} is. Further insertions give more powers of $1/z$ which multiply general matrices in (ab) space. Thus, the “Lorentz” symmetry guarantees that the amplitude has the form

$$M^{ab} = (cz + \dots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \dots \quad (2.16)$$

where A^{ab} is antisymmetric in (ab) .

We can now find the z -dependence of the amplitude for various helicity combinations by contracting our ansatz for M^{ab} with polarization vectors. The Ward identity further constrain M^{ab} . For Yang-Mills theory, the Ward identity says that

$$p_{ja}(z)M^{ab}\epsilon_{kb} = 0, \quad (2.17)$$

and similarly with j and k reversed, (but recall that $p_{ja}M^{ab} \neq 0$ when the second b index is not contracted with ϵ_k). This implies

$$p_{ja}(z)M^{ab}\epsilon_{kb\nu} = 0 \quad \implies \quad q_a M^{ab}\epsilon_{kb} = -\frac{1}{z}p_{1a}M^{ab}\epsilon_{kb} \quad (2.18)$$

which is extremely useful because $\epsilon_1^- = q$, so we can use it to replace $\epsilon_j^- \rightarrow -\frac{1}{z}p_j$. Using this information, let us look at the large z amplitudes for a few helicity combinations. Consider first M^{-+} ; recall that this was the only case that was understood by BCFW directly in terms of Feynman diagrams

$$\begin{aligned} M^{-+} &= \epsilon_{ja}^- M^{ab}\epsilon_{kb}^+ = -\frac{1}{z}p_{ja} \left[(cz + \dots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \dots \right] q_b \\ &= -\frac{1}{z}p_{ja}A^{ab}q_b + O(1/z^2) \rightarrow \frac{1}{z} \end{aligned} \quad (2.19)$$

as $z \rightarrow \infty$. A more non-trivial case is

$$\begin{aligned} M^{--}(z) &= \epsilon_{ja}^- M^{ab}\epsilon_{kb}^- = -\frac{1}{z}p_{ja} \left[(cz + \dots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \dots \right] (q_b^* + zp_{jb}) \\ &= -\frac{1}{z}p_{ja}A^{ab}q_b^* - \frac{1}{z}p_{ja}B^{ab}p_{jb} + O(1/z^2) \rightarrow \frac{1}{z} \end{aligned} \quad (2.20)$$

as $z \rightarrow \infty$; note that we have used the fact that $q^* \cdot p_j$, p_j^2 , and $p_{ja}p_{jb}A^{ab}$ all vanish, the last due to the anti-symmetry of A^{ab} . As a last example before we simply list the results, let us consider

$$\begin{aligned} M^{+-}(z) &= \epsilon_{ja}^+ M^{ab}\epsilon_{kb}^- = (q_a^* - zp_{ka}) \left[(cz + \dots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \dots \right] (q_b^* + zp_{jb}) \\ &\rightarrow z^3. \end{aligned} \quad (2.21)$$

$\epsilon_1 \backslash \epsilon_2$	-	+	T
-	$1/z$	$1/z$	$1/z$
+	z^3	$1/z$	z
T1	z	$1/z$	z
T2	z	$1/z$	1

Table 1: This table displays the results for the z -scalings in gauge theory. The difference between T2 and T1 is simply whether or not $\epsilon_i^T \cdot \epsilon_j^T = 0$, respectively.

Table 1 displays the general results for gauge theory. The difference between T2 and T1 is simply whether or not $\epsilon_j^T \cdot \epsilon_k^T = 0$, respectively. We have checked these results for the $2 \rightarrow 2$ amplitude in any number of dimensions by using results in the literature. Since $M(z \rightarrow \infty)$ vanishes for all the $(-, h)$ helicity combinations, BCFW recursion relations can be used to compute tree gluon amplitudes in any number of dimensions.

Note that for this discussion, we did not use the q -lightcone gauge for the background; exploiting this gauge gives us further information. As before, q -lightcone gauge eliminates the $O(z)$ vertices except for the unique set of diagrams — but as with the scalar-YM case, these diagrams only exist if the hard momentum lines are adjacent in color. This is also true of sub-leading vertices coming from a single insertion of the F^{ab} interaction. Therefore, for non-adjacent colors, M^{ab} begins at $O(1/z)$. This is enough to guarantee that e.g. M^{-+} scales as $1/z^2$ and not $1/z$, and that M^{+-} scales as z^2 rather than z^3 , as can again be confirmed for explicit $2 \rightarrow 2$ scattering amplitudes. The q -lightcone gauge will play a more important role in controlling the large z amplitudes in gravity below.

2.5 Scalar-graviton amplitudes

Now we consider gravity, beginning with the case two scalar-graviton amplitudes, where we analytically continue the scalar momenta. The Lagrangian is

$$L = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (2.22)$$

Naively, there are now z^2 and z vertices, and the amplitudes should blow up with increasing powers of z for more graviton legs. However, using diffeomorphism invariance we can choose light-cone gauge for the background $g^{\mu\nu}$; taking q_μ to point in the $+$ direction the gauge choice is

$$g^{++} = g^{+i} = 0 \quad \text{and} \quad g^{+-} = 1. \quad (2.23)$$

Note that equivalently, writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the gauge choice is $h_{-\mu} = 0$.

Light-cone gauge eliminates the $O(z^2)$ vertices, but again we have our unique diagram when the two external scalar lines couple to a single insertion of the background $g^{\mu\nu}$ with an $O(z^2)$ vertex. Thus

$$M(z) \rightarrow z^2. \quad (2.24)$$

This is much better behaved than the naive expectations, reflecting the power of exploiting background light-cone gauge for gravity amplitudes, although since $M(z)$ still diverges there are no BCFW recursion relations using analytically continued scalar momenta.

2.6 Photon-graviton amplitudes

We now move on to consider two photon-graviton amplitudes where we analytically continue the photon momenta, corresponding to studying a hard photon moving in a soft gravitational background. Our experience with gauge theory suggests that we exploit an enhanced spin symmetry at infinite momentum. To make this manifest, we need to introduce Lorentz (rather than space-time) indices. For this purpose, we use the vielbein e_μ^a ; this introduces, in addition to the usual diffeomorphism redundancy, a gauge Lorentz redundancy acting on the a indices, with the associated connection ω_μ^{ab} . The standard relation to the metric variables are

$$g_{\mu\nu} = e_{\mu a} e_{\nu b} \eta^{ab}, \quad \omega_{\mu ab} = e_a^\nu \nabla_\mu e_{\nu b}. \quad (2.25)$$

Since these fields connect the asymptotic Lorentz tensor structure to the local geometry, they are exactly what we need. The Lorentz gauge redundancy is useful; in addition to fixing the diffeomorphism redundancy for the metric by choosing metric light-cone gauge, we will fix the extra Lorentz redundancy by fixing light-cone gauge for $\omega_{\mu ab}$.

Let us now consider the Lagrangian for a photon in a gravitational background

$$L = -\frac{1}{4} \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} \nabla_{[\mu} A_{\nu]} \nabla_{[\alpha} A_{\beta]}. \quad (2.26)$$

If we add the gauge fixing term $(\nabla_\mu A^\mu)^2$, we obtain

$$L = -\frac{1}{2} \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta \quad (2.27)$$

where we have dropped a term proportional to $R^{\mu\nu}$ because it vanishes on the background field equations. We introduce the vielbein so that

$$A_\mu = e_\mu^a A_a, \quad \nabla_\nu A_\mu = e_\mu^a D_\nu A_a, \quad \text{with } D_\nu A_a = \partial_\nu A_a + \omega_{\nu a}^c A_c \quad (2.28)$$

The Lagrangian becomes

$$L = -\sqrt{-g} g^{\mu\nu} \eta^{ab} (\partial_\mu A_a + \omega_{\mu a}^c A_c) (\partial_\nu A_b + \omega_{\nu b}^d A_d). \quad (2.29)$$

Note that again, the two-derivative interactions which dominate the amplitude at large z respect a spin ‘‘Lorentz’’ invariance, broken by the subleading interactions through non-vanishing $\omega_{ab\mu}$ which is antisymmetric in (ab) . Choosing light-cone gauge for both $g^{\mu\nu}$ and $\omega_{\mu ab}$,

$$g^{++} = g^{+i} = 0, \quad g^{+-} = 1 \quad \text{and } \omega_{ab}^+ = 0, \quad (2.30)$$

we see that there are no $O(z^2)$ vertices and the only $O(z)$ vertices preserve the spin Lorentz invariance; except for the by now familiar unique set of diagrams. Thus $O(z^2)$ interactions

$\epsilon_1 \backslash \epsilon_2$	-	+	T
-	1	1	1
+	z^4	1	z^2
T1	z^2	1	z^2
T2	z^2	1	z

Table 2: This table displays the z -scalings for a photon in a gravitational field.

must be proportional to η^{ab} , while the $O(z)$ interactions not proportional to η^{ab} must involve a single insertion of the connection $\omega_{\mu ab}$ which is anti-symmetric in (ab) and so gives an anti-symmetric contribution to M^{ab} . We therefore find

$$M^{ab} = cz^2\eta^{ab} + zA^{ab} + B^{ab} + \dots \tag{2.31}$$

where A^{ab} , like ω , must be antisymmetric, and B^{ab} is arbitrary. Using the Ward Identity as in the Yang-Mills section, the large z behavior of the amplitude for all helicity combinations can be computed, and are displayed in table 2. We have checked that these results agree with the known amplitudes for the $2 \rightarrow 2$ graviton-photon scattering amplitudes — a non-trivial check, since individual Feynman diagrams diverge at least as fast as z^2 , with possible additional z 's coming from contraction with the polarization vectors.

2.7 Amplitudes in general relativity

We finally apply the lessons above to prove the BCFW Recursion Relations for graviton amplitudes. Of course the results can be anticipated via the KLT relations which express graviton amplitudes as products of Yang-Mills amplitudes $M_{grav} \sim M_{gauge} \times M_{gauge}$. Indeed we will use a simple and natural trick [23] that was originally developed to help make the KLT relations manifest in GR, in order to manifest an even larger spin “Lorentz” invariance for graviton amplitudes, which will determine the large z behavior we seek.

The quadratic lagrangian for a gravitational fluctuation $h_{\mu\nu}$ about an arbitrary background $g_{\mu\nu}$ is, after adding the standard background de-Donder gauge fixing term [24]

$$L = \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} \nabla_\mu h_\alpha^\beta \nabla_\nu h_\beta^\alpha - \frac{1}{8} g^{\mu\nu} \nabla_\mu h_\alpha^\alpha \nabla_\nu h_\beta^\beta - h_{\alpha\beta} h_{\mu\nu} \frac{1}{2} R^{\beta\mu\alpha\nu} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] \tag{2.32}$$

where we have used to the background field equations to set $R_{\mu\nu} = 0$, and we have included an extra dilaton field ϕ .

Since dilaton number is conserved, at tree level the dilaton will decouple from amplitudes involving only spin-2 graviton external states. We have included it anyway because it will allow us to perform a field re-definition that eliminates the h_α^α terms in the Lagrangian, which will enable us to manifest two separate copies of a spin “Lorentz” invariance acting separately on the left and right indices of h . The field redefinition is

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + g_{\mu\nu} \sqrt{\frac{2}{D-2}} \phi, \quad \phi \rightarrow \frac{1}{2} g^{\mu\nu} h_{\mu\nu} + \sqrt{\frac{D-2}{2}} \phi \tag{2.33}$$

so that the lagrangian simply becomes

$$L = \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} \nabla_\mu h_{\alpha\beta} \nabla_\nu h_{\rho\sigma} - \frac{1}{2} h_{\alpha\beta} h_{\mu\nu} R^{\beta\mu\alpha\nu} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]. \quad (2.34)$$

We will henceforth drop the (re-defined) dilaton field, since it decouples from the amplitudes we are interested in.

Mirroring the photon-graviton analysis above, we will now introduce a vielbein for the background field. In order to make a clear distinction between ‘left’ and ‘right’ indices, we will use a ‘left’ vielbein e and a ‘right’ vielbein \tilde{e} , which introduces two copies of Lorentz gauge redundancies with their associated connections $\omega, \tilde{\omega}$. We then write

$$h_{\mu\nu} = e_\mu^a \tilde{e}_\nu^{\tilde{a}} h_{a\tilde{a}}, \quad \nabla_\alpha h_{\mu\nu} = e_\mu^a \tilde{e}_\nu^{\tilde{a}} D_\alpha h_{a\tilde{a}} \quad (2.35)$$

with

$$D_\alpha h_{a\tilde{a}} = \partial_\alpha h_{a\tilde{a}} + \omega_{\alpha a}^b h_{b\tilde{a}} + \tilde{\omega}_{\alpha \tilde{a}}^{\tilde{b}} h_{a\tilde{b}}. \quad (2.36)$$

Of course in reality $e = \tilde{e}$ and $\omega = \tilde{\omega}$, but this simple notation will help to keep track of the fact that left-right index contractions never occur. The lagrangian becomes

$$L = \sqrt{-g} \left[\frac{1}{4} g^{\mu\nu} \eta^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}} D_\mu h_{a\tilde{a}} D_\nu h_{b\tilde{b}} - \frac{1}{2} h_{a\tilde{a}} h_{b\tilde{b}} R^{ab\tilde{a}\tilde{b}} \right]. \quad (2.37)$$

As in the photon-graviton case above, we choose light-cone gauge so that

$$\omega_{ab}^+ = \tilde{\omega}_{\tilde{a}\tilde{b}}^+ = g^{++} = g^{+i} = 0 \quad \text{and} \quad g^{+-} = 1, \quad (2.38)$$

and there are no $O(z^2)$ vertices and the only $O(z)$ vertices preserve both the spin Lorentz symmetries — except for the unique set of diagrams that give contributions up to $O(z^2)$. The $O(z^2)$ terms must come from the two derivative part of the lagrangian, which don’t break either of the left or right spin “Lorentz” invariances, and are thus proportional to $\eta^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}}$. The $O(z)$ terms that violate the symmetry come from a derivative on h and a single ω or $\tilde{\omega}$ insertions, and hence have the form $\eta^{ab} \tilde{A}^{\tilde{a}\tilde{b}} + A^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}}$ where A, \tilde{A} are antisymmetric. Now consider the $O(1)$ parts of the amplitude. Since all propagators scale as $1/z$, these can only come directly from the $O(1)$ vertices in the Lagrangian. There are terms involving ω^2 or $\tilde{\omega}^2$, each of which breaks one of the “Lorentz” symmetries but not the other, so these give amplitudes of the form $\eta^{ab} B^{\tilde{a}\tilde{b}} + B^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}}$. There are also terms proportional to $\omega \tilde{\omega}$ and the Riemman tensor, which are antisymmetric separately in (ab) and $(\tilde{a}\tilde{b})$, and which thus give a contribution to the amplitude $A^{ab\tilde{a}\tilde{b}}$ which has the same antisymmetry properties. Thus we find

$$M^{a\tilde{a}b\tilde{b}} = cz^2 \eta^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}} + z \left(\eta^{ab} \tilde{A}^{\tilde{a}\tilde{b}} + A^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}} \right) + A^{ab\tilde{a}\tilde{b}} + \eta^{ab} \tilde{B}^{\tilde{a}\tilde{b}} + B^{ab} \tilde{\eta}^{\tilde{a}\tilde{b}} + \frac{1}{z} C^{ab\tilde{a}\tilde{b}} + \dots \quad (2.39)$$

where A^{ab} is an antisymmetric matrix, B^{ab} is an arbitrary matrix, and $A^{ab\tilde{a}\tilde{b}}$ is antisymmetric in (ab) and $(\tilde{a}\tilde{b})$. It is quite remarkable that this symmetry structure is precisely what we would get from ‘squaring’ the Yang-Mills ansatz above — for instance the $\eta^{ac} \tilde{B}^{\tilde{b}\tilde{d}}$ type

$\epsilon_1 \backslash \epsilon_2$	--	-+	++	-T	+T	TT
--	$1/z^2$	$1/z^2$	$1/z^2$	$1/z^2$	$1/z^2$	$1/z^2$
-+	z^2	z^2	$1/z^2$	z^2	1	1
++	z^6	z^2	$1/z^2$	z^4	1	z^2
-T	1	1	$1/z^2$	1	1 or $1/z$	1 or $1/z$
+T	z^4	z^2	1	z^4 or z^3	1	z^2 or z
TT	z^2	1	$1/z^2$	z^2 or z	1 or $1/z$	$z^2, z, \text{ or } 1$

Table 3: This table displays the z -scalings for gravity amplitudes, and more generally, for two-index tensor fields. The various possibilities involving T polarizations depend on whether or not the T factors in the graviton polarization tensors are parallel or orthogonal.

terms come from multiplying the $z\eta^{ab}$ pieces in M_{gauge}^{ab} with the $(1/z)B^{\tilde{a}\tilde{b}}$ terms in $M_{\text{gauge}}^{\tilde{a}\tilde{b}}$, while the symmetry structure of $A^{ab\tilde{a}\tilde{b}}$ arises from the product of the two anti-symmetric matrices, $A^{ab}\tilde{A}^{\tilde{a}\tilde{b}}$.

Having established this ansatz for $M^{ab\tilde{a}\tilde{b}}$, we contract with the graviton polarization tensors to obtain the physical amplitudes. We again use the Ward identity to further simplify the amplitude. The identity says that

$$p_{ja}(z)M^{a\tilde{a},b\tilde{b}}\epsilon_{kb\tilde{b}} = 0, \tag{2.40}$$

so as in the gauge case, we can use this to show that

$$(p_{ja} + zq_a)M^{a\tilde{a},b\tilde{b}}\epsilon_{kb\tilde{b}} = 0 \implies q_a M^{a\tilde{a},b\tilde{b}}\epsilon_{kb\tilde{b}} = -\frac{1}{z}p_{ja}M^{a\tilde{a},b\tilde{b}}\epsilon_{kb\tilde{b}}. \tag{2.41}$$

This means that we can take

$$\epsilon_{ja\tilde{a}}^- = q_a q_{\tilde{a}} \rightarrow \frac{1}{z^2}p_{ja}p_{j\tilde{a}} \tag{2.42}$$

when we compute $(--, h)$ amplitudes. A quite non-trivial example is

$$\begin{aligned} M^{--,--}(z) &= \epsilon_{ja\tilde{a}}^- M^{a\tilde{a},b\tilde{b}} \epsilon_{kb\tilde{b}}^- = \frac{1}{z^2}p_{ja}p_{j\tilde{a}} M^{a\tilde{a},b\tilde{b}}(q_b^* + zp_{jb})(q_{\tilde{b}}^* + zp_{j\tilde{b}}) \\ &= \frac{1}{z}C^{ab\tilde{a}\tilde{b}}p_{ja}p_{j\tilde{a}}p_{jb}p_{j\tilde{b}} + O\left(\frac{1}{z^2}\right) \rightarrow \frac{1}{z} \end{aligned} \tag{2.43}$$

as $z \rightarrow \infty$. This is good enough to obtain recursion relations, though a little extra work shows that $C^{ab\tilde{a}\tilde{b}}$ is not a completely generic tensor but is a sum of terms antisymmetric in (ab) , and in $(\tilde{a}\tilde{b})$, so that even the $O(1/z)$ term above vanishes and the leading amplitude scales as $1/z^2$. Other results are similar and, as we noted above, they conform to the pattern $M_{\text{grav}} \sim M_{\text{gauge}} \times M_{\text{gauge}}$. We have checked that these accord with behavior of the $2 \rightarrow 2$ gravitational scattering amplitudes in arbitrary D . Since $M(z \rightarrow \infty)$ vanishes for all $(--, h)$ helicity combinations, the BCFW Recursion Relations hold in general relativity for all dimensions $D \geq 4$.

3. Discussion

We close with brief comments on some possible implications of our results. The ability to use BCFW recursion relations to compute higher-dimensional amplitudes can be useful for computing certain massive 4D amplitudes where the massive particles can be thought of as KK modes in the dimensional reduction of the higher-dimensional theory (other extensions of recursion relations to include massive particles have been discussed in [25, 26]). This can be used for the analytic computation of some massive SM amplitudes of relevance to the LHC. For instance, to compute $gg \rightarrow t\bar{t} + ng$, we can consider a 5D amplitude with all massless particles where all the gluon momenta are four-dimensional but the 5D top quarks carry five-momentum. However, while it is nice to have analytic expressions for these amplitudes, it does not really appreciably help with bread and butter QCD physics, as the amplitudes can be numerically determined in any case. The real bottleneck is not in determining amplitudes at tree level, but in performing the phase space integrals needed to convert the amplitudes to rates. Nevertheless, our discussion does suggest interesting avenues for further theoretical exploration.

Our analysis of the large z scaling of tree amplitudes relied heavily on the form of the Yang-Mills and Gravity Lagrangians. However we know that the structure of Yang-Mills and Gravity are forced on us by consistent S-matrices for massless spin 1 and spin 2 particles (with minimal derivative interactions). It must therefore be possible and illuminating to determine the large z scalings directly from S-matrix arguments, without passing through the Lagrangian as an intermediary.

While the BCFW recursion relations beautifully realize the S-matrix program for tree amplitudes, the situation at loop level is not quite as transparent even though much is understood [27]. One issue, for instance, is the analytic structure of $M(z)$, where in addition to expected poles and cuts, there are also ‘unreal’ poles without a clear physical interpretation. It would be interesting to see if our picture sheds any further light on this. Furthermore, the large z scaling of amplitudes is modified at loop level but, in examples, continues to exhibit very interesting patterns that would be interesting to understand along the same lines as our tree-level analysis.

A related issue is the much better than expected behavior that has been found for gravity amplitudes at loop level. The most intriguing recent example of this has been for $N = 8$ supergravity, where cancelations not obviously guaranteed by SUSY have led some to conjecture that the theory might even be perturbatively finite [28, 29]. But even pure gravity amplitudes appear to be better behaved than expected by naive power-counting [15, 16]. This seems very surprising, especially since in the usual view, power-counting is controlled purely by Wilsonian dimensional analysis, and does not care about whether we have a “simple” non-renormalizable theory of scalars like the chiral Lagrangian, or a more “complicated” one such as gravity. However it appears that precisely these “complicated” theories might have unexpectedly good UV behavior! Why should this be?

A possible clue is that these cancelations have progenitors in the soft large z behavior of tree amplitudes we have discussed in this paper [15], which arise as cuts of loop diagrams. But we have understood why certain graviton amplitudes are softer at infinite (complex)

momentum than gauge amplitudes which are in turn softer than the scalar ones — at infinite momentum, the amplitudes for spin s particles are governed by s copies of spin “Lorentz” symmetries — so the more “complicated” theories with higher spin are actually constrained by larger symmetries. It is tempting to speculate that these enhanced symmetries are further extended in theories with high degrees of supersymmetry and can help illuminate the mysterious cancellations found in $N = 8$ supergravity.

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